JOURNAL OF THE EXPERIMENTAL ANALYSIS OF BEHAVIOR 2006, **85,** 425–435 NUMBER 3 (MAY)

NOTES ON DISCOUNTING

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In general, if a variable can be expressed as a function of its own maximum value, that function may be called a discount function. Delay discounting and probability discounting are commonly studied in psychology, but memory, matching, and economic utility also may be viewed as discounting processes. When they are so viewed, the discount function obtained is hyperbolic in form. In some cases the effective discounting variable is proportional to the physical variable on which it is based. For example, in delay discounting, the physical variable, delay (D), may enter into the hyperbolic equation as kD. In many cases, however, the discounting data are not well described with a single-parameter discount function. A much better fit is obtained when the effective variable is a power function of the physical variable (kD^s in the case of delay discounting). This power-function form fits the data of delay, probability, and memory discounting as well as other two-parameter discount functions and is consistent with both the generalized matching law and maximization of a constant-elasticity-of-substitution utility function.

Key words: discounting, delay discounting, memory discounting, social discounting, probability discounting, matching, rational behavior, utility maximization

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According to *The Oxford Encyclopedic English Dictionary* (Hawkins & Allen, 1991), one meaning of the verb *discount* is to ''…reduce the effect (of an event, etc.) by previous action.'' Another meaning is to ''detract from; lessen; deduct…'' The term is used most commonly in economic interchange. For instance, if you buy large amounts of a commodity the seller may discount its price. However, the term often is used in a broader context; for instance (from a television review in *The New York Times*): ''There is always some way of trying to discount what is fundamentally disturbing about a great work of art'' (Smith, 2005, p. E1). In the present article the term will be used broadly to mean the reduction of a quantity with the increase of some variable. The article's purpose is not to present a single theory of discounting but merely to examine similarities and differences among several kinds of quantifiable discounting processes.

In the general case, a quantity, X, is reduced by some variable to a lesser quantity, x:

$$x = \pi X \tag{1}$$

$$\frac{x}{X} = \pi \tag{1a}$$

where X is the original quantity, π is the discounting variable, and x is the discounted quantity. If you were buying cereal at 50% off list price, for example, X would be the list price, x would be the price you pay, and π (= 0.5) would be the discount.

The discounting variable π must always be a fraction ($0 \le \pi \le 1$) otherwise there would be no reduction in X. The symbol π was used by Kahneman and Tversky (1979) for ''decision weight,'' the discounting variable in their theory of probability discounting (prospect theory). Here, the symbol is meant to apply to all forms of discounting. When $\pi = 1$, there is no discounting and $x = X$; as π decreases, x decreases proportionally; when $\pi = 0$, discounting is complete. In most cases of discounting, π will be some function of a physical variable or a combination of physical variables. That function is called a *discount function.* In the case of price discounting, π might be a function of volume bought or other economic variables such as time on the shelf and competitors' prices. Table 1 lists a sampling of kinds of discounting in terms of Equation 1.

Equation 1, which expresses the absolute value of x as a function of the absolute value of the discounted variable, X, is the usual form in which such equations are written. Equation 1a,

This article was written with the support of a grant from the National Institutes of Health.

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doi: 10.1901/jeab.2006.85-05

Table 1

Kinds of discounting. The symbol f^{-1} indicates that π would be expected to vary inversely with the variable in parenthesis.

Delay of Reward	X = absolute reward value
	x = current value
	$\pi = f^{-1}$(delay to reward)
Memory	X = original learning
	x = memory
	$\pi = f^{-1}$(time between learning and recall)
Probability of Reward	X = absolute reward value
	x = value of probabilistic reward
	π = f(probability of reward)
Generosity	X = money you have
	x = money you give to another person
	$\pi = f^{-1}$(social distance from that person)
Energy	X = energy of source (e.g. light, sound)
	x = energy distant from source
	$\pi = f^{-1}$(distance from source)

the normalized form, reflects the fact that, with regard to choice, only relative value has meaning (Rachlin, 1971). Moreover, Equation 1a may be used to compare discounting of different commodities and in other situations where the absolute value of X may be difficult or impossible to establish. Henceforth, except as noted, we will use the normalized form.

Magnitude Discounting. The problem with Equation 1 or 1a as a general discounting formula is that, except for probability itself (to be discussed later), the variables that cause discounting are positive, unlimited magnitudes—measurable on ordinal, interval, or ratio scales—that start at zero and increase indefinitely. For the sake of brevity I will call them, simply, *magnitudes* as distinct from *probabilities*—that start at unity and decrease to zero. For example, the value of a reward is discounted by the delay (D) to its receipt. In the case of delay, the point of no discounting ($D = 0$) corresponds to a probabilistic discount of unity ($\pi = 1$) whereas complete discounting ($D \rightarrow \infty$) corresponds to a probabilistic discount of zero ($\pi \rightarrow 0$). We need to convert a magnitude that begins at zero and may increase indefinitely, to a fraction—a variable that begins at unity and descends to zero. A straightforward transformation would be:

$$\pi = \frac{1}{1 + \delta} \qquad (2)$$

The variable δ ($\delta \geq 0$) determines degree of discounting. When $\delta = 0$, $\pi = 1$. As δ increases indefinitely, π decreases to zero. Equation 2 is a general form of the equation by which the magnitude, odds against, is related to the fraction, probability. (We will discuss probability discounting later.) But δ is dimensionless. Most actual magnitudes (such as time and distance) are measured in specific units (seconds, inches, etc.). Let us symbolize any such magnitude (time, distance, etc.) as φ. Then:

$$\delta = k\varphi \qquad (3)$$

Since δ is dimensionless, the dimension of k must be the reciprocal of the dimension of φ. Substituting in Equation 2 and then in Equation 1a:

$$\frac{x}{X} = \frac{1}{1 + k\varphi} \qquad (4)$$

When $\varphi = 0$, there is no discounting and $x/X = 1$; as φ increases, x/X decreases hyperbolically; as φ increases infinitely, x/X approaches 0. Equation 4 is a generalized version of hyperbolic discounting. But Equation 4 is just a simple transformation of Equation 1 to accommodate a magnitude that ranges from zero to infinity.

Note that the addition of 1 to $k\varphi$ is not just a convenience to deal with abnormally low values of φ but is basic to the transformation of a magnitude (such as odds against) to a fraction (such as probability). In physics, for example, the energy at a point on a wave propagating in two dimensions (like a tsunami) dissipates over the perimeter of an ever-expanding circle. Since the perimeter of a circle increases proportionally to its radius, the energy at a point on the wave decreases proportionally to the distance from the source according to Equation 4, where the fraction x/X measures the energy at a distance from the source relative to the energy at the source (assuming a perfectly circular energy spread). The distance, φ, is a magnitude, as I have defined it, and is converted to a fraction by Equation 4.

In the case of delay discounting, φ corresponds to delay (D) and Equation 4 becomes Mazur's (1987) hyperbolic delay discounting equation:

$$\frac{v}{V} = \frac{1}{1 + kD} \qquad (5)$$

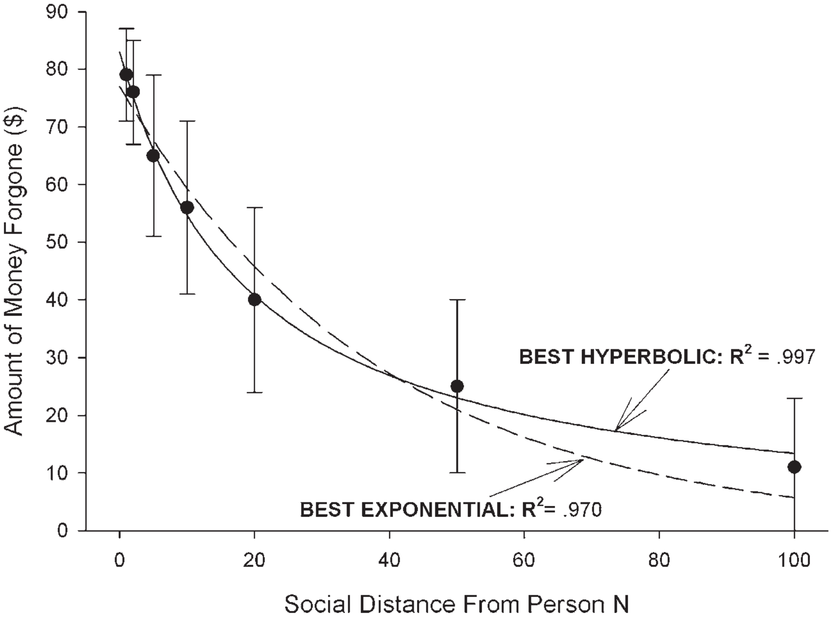

Figure 1. Social discounting. Amount of money forgone as a function of social distance from the receiver. The solid curve is the best-fitting simple hyperbolic discount function (Equation 4). The dashed curve is the best-fitting exponential discount function (Equation 6).

where V is the undiscounted value of a reward, v is its discounted value, and D is its delay. In many cases of psychological discounting, Equation 4 is the form empirically obtained. Mazur (1987) found that it described pigeons' choices between food rewards of different amounts and delays. Exponentiating the denominator did not significantly increase the fit of Equation 5 to the data.

Recently, Jones and Rachlin (2006) found that Equation 4 precisely described the average results of more than 300 human participants who each chose whether to share a hypothetical amount of money with another person at a greater or lesser social distance; the discounting variable, social distance (N), was measured as numerical order in closeness, to the participant, of the person who would be sharing the money (number 1 being the closest, number 2 being the second closest, and so forth). Participants were asked to imagine that they had made a list of the 100 people closest to them in the world ranging from their dearest friend or relative at number 1 to (possibly) a mere acquaintance at number 100. Then they were given a series of two-column lists. Column A listed dollar amounts ranging from $75 to $155 in $10 increments "for you alone" (the "selfish" option); for each column-A amount, column B was always the same alternative repeated: "$75 for you and $75 for the [*Nth*] person on the list" (the "generous" option). Participants chose between A and B for each item on each list. The social distance from the participant to the recipient (N) was constant on each list but varied between lists. The dependent variable was the crossover point between the "generous" alternative (preferred with low column-A amounts) and the "selfish" alternative (preferred with high column-A amounts). This crossover point represents the maximum amount of money the participant was willing to forgo in order to give $75 to person N.

We found that the greater the social distance of the receiver from the sharer (N), the less money the sharer was willing to forgo. That is, "generosity" was discounted by social distance. Figure 1 shows the average discount function with social distance (N) taking the place of φ in Equation 4. Social distance as thus measured is an ordinal scale, not a ratio scale. Nevertheless, it has the property of beginning at zero (the participant himself or herself) and increasing indefinitely with perceived social distance from the (hypothetical) receiver. In Figure 1, the solid line is Equation 4 (absolute rather than normalized version) with x = money forgone, X = \$83 (the best-fitting value at $N = 0$), and $k = 0.052$. The variance accounted for (R^2) is .997.

Rachlin (1989) and Rachlin and Raineri (1992) speculated that social discounting would be found to correspond to temporal discounting, and hence that social cooperation would correspond to self-control. As Ainslie (1992) pointed out, if you imagine a person now, in time, as corresponding to a person here, in social space, and that same person in future times as corresponding to other people in social space, then learning self-control (i.e., to overcome impulsiveness) would correspond to learning social cooperation (i.e., to resist selfishness). Of course, the formal correspondence between the two discounting processes is just that and not proof that a common causal mechanism exists or that one process underlies the other.

Simple and compound interest. In economics, Equation 4 is used to calculate simple interest. Let x be the principal, k be the interest rate, φ be the term of the loan, and X be the payout (principal plus interest). Equation 4, solving for X [$X = x(1 + k\varphi)$], expresses payout as a function of principal, interest rate, and term.

Compound interest is derived from simple interest and is used in cases where the term (φ) may be unknown. With compounding, simple interest is calculated over some fixed period, t, added to the principal, and repeated at intervals of t. For example, if you deposit money in the bank you may not know when you will want to withdraw it. If, when you did come to withdraw your money, the bank then calculated simple interest from the time of deposit, you would have an incentive, after a short period of time, to withdraw your money plus the interest and deposit it in another bank, thus compounding the interest yourself. So as not to lose your account in this way (and because you might compound it yourself anyway by withdrawing and then redepositing your money), the bank compounds your interest for you. As the period of compounding (t) approaches zero, the resulting overall discount function approaches the exponential function:

$$\frac{x}{X} = e^{-i\varphi} \tag{6}$$

where X is the balance for an original deposit of x after a time, φ, and with an interest rate, i. For exponential delay discounting, x is the discounted value of X, φ = delay, and interest rate (i) is a constant representing degree of discounting, corresponding to k in Equation 4.

That is, the hyperbolic discount formula corresponds to the *simple* interest formula in economics whereas the exponential discount formula corresponds to the *compound* interest formula (derived from simple interest).

Power-Function Discounting. Stevens's (1957) power law says that a psychologically effective variable (a "sensation" in psychophysics) is a power function of its physical cause. In the present case:

$$\delta = k\varphi^s \tag{7}$$

where δ is the sensation, φ is the physical stimulus, and k and s are constants. The constant k ($k \geq 0$) is a scaling constant and may differ widely across individuals; the exponent s ($s \geq 0$) measures the sensitivity of the sensation to the physical stimulus; when $s = 0$, δ is completely insensitive to variation in φ; as s increases, sensitivity increases and, in psychophysical experiments, has been found to be fairly constant across individuals but to vary with the modality of the physical variable. For example, in psychophysical magnitude-estimation experiments, the exponent for brightness as a function of light intensity is 0.33 whereas the exponent for loudness as a function of sound intensity is 0.67 (Luce & Krumhansl, 1988).

Substituting in Equation 2 and then in Equation 1a:

$$\frac{x}{X} = \frac{1}{1 + k\varphi^s} \tag{8}$$

The physical stimulus in Equation 7 (φ) becomes the physical discounting variable in Equation 8. When $\varphi = 0$ there is no discounting and $x/X = 1$; as φ increases, x/X decreases hyperbolically; as φ increases infinitely, x/X approaches 0.

Equation 8 is the more general formula and Equation 4 is the particular case where the discounting exponent s equals 1.0. But $s = 1.0$ is not just an arbitrary point. As we have seen, some delay discounting studies and social discounting studies have found unitary discounting exponents. Moreover, the psychophysical exponents for both subjective time and subjective distance, the usual discounting variables, are both close to unity (Luce & Krumhansl, 1988). It is at least conceivable that the frequently found deviations from s = 1.0 in Equation 8 eventually will be explicable in terms of other factors than simple time and distance judgments—for examples, the expectation of performing anticipatory activities during the delay period, differences of constraints on reward consumption (Raineri & Rachlin, 1993), or differences in valence or quality of outcome—whether it is a reward or punisher and what kind of reward or punisher it is (Estle, Green, Myerson & Holt, in press). Discounting variables may differ with all of these variables while absolute delays remain the same.

It should be said, however, that Equation 8, proposed by Rodriguez and Logue (1988) as a general discount function, is not the typical form by which Equation 4 has been generalized. The usual tactic, proposed by Rachlin (1989, p. 136) and by Loewenstein and Prelec (1992), has been to exponentiate the entire denominator as in Equation 9:

$$\frac{x}{X} = \frac{1}{(1 + k\varphi)^s} \tag{9}$$

Equation 9, like Equation 8, is consistent with Stevens's psychophysical law (Myerson & Green, 1995). Moreover, Green and Myerson (2004) have found Equation 9 to fit a wide range of both probability and delay discount data. However, on the basis of variance accounted for (R^2) alone, it is not possible to choose between Equations 8 and 9. Raineri (1991) obtained both probability and delay discount functions using a method described by Raineri and Rachlin (1993) with differing reward amounts varying over a wide range. Table 2 shows values of the constants k and s and the goodness of fit for probabilistic and delay discounting with hypothetical rewards of $100, $10,000, and $1,000,000. Variance accounted for is very high with both equations. The average probability-discounting R^2 across the three reward amounts is .993 for Equation 8 and .989 for Equation 9. The average delay-discounting R^2 is .995 for both equations. One advantage of Equation 9 is that it is easily differentiable. Also, as Loewenstein and Prelec (1992) pointed out, if s in Equation 9 is set as s'/k, and k goes to zero, then Equation 9 becomes exponential (Equation 6 with $i = s'$). The main advantage of Equation 8 will be evident later in the article when we rewrite the equations underlying both matching and maximizing as functions of their maximum values (i.e., as discount functions). When this is done, both matching and maximizing take the form of Equation 8.

Table 2

Values of the constants k and s and goodness of fit to group medians for Equations 8 and 9 for data from Raineri (1991) with both delay and probability discounting.

	Equation 8	Equation 9
Delayed Amounts		
$100	k = .718	k = 1.44
	s = .781	s = .622
	R^2 = .997	R^2 = .994
$10,000	k = .276	k = .219
	s = .963	s = 1.11
	R^2 = .991	R^2 = .991
$1,000,000	k = .083	k = .017
	s = 1.43	s = 7.10
	R^2 = .997	R^2 = .999
Probabilistic Amounts		
$100	k = 1.19	k = 3.51
	s = .720	s = .550
	R^2 = .988	R^2 = .976
$10,000	k = 2.81	k = 2.28
	s = 1.00	s = 1.14
	R^2 = .996	R^2 = .997
$1,000,000	k = 5.81	k = 6.87
	s = .944	s = .954
	R^2 = .995	R^2 = .994

Following Baum (1974), we say that the exponent s in Equation 8 indicates *sensitivity* of x/X to φ. Baum's original discussion of sensitivity was in the context of a generalized version of Herrnstein's (1961) matching law, to be discussed later. However, as with matching, when the exponent s is zero, the dependent variable x/X is invariant with the independent variable φ. As the exponent increases, x/X becomes a steeper and steeper function of φ.

In physics, when energy of any kind (or gravity or an electrical charge) is propagated in three dimensions (as opposed to the two-dimensional wave previously mentioned), it spreads out over the surface of a sphere (as opposed to the perimeter of a circle). The surface area of a sphere increases as the *square* of its radius. Light energy from a candle therefore decreases as the square of its distance from the candle:

$$\frac{e}{E} = \frac{1}{1 + kr^2} \quad (10)$$

where e/E is the energy measured at a point at a distance r from the source E relative to that at the source. Equation 10 is a form of the well-known inverse square law.[1]

Probability discounting. When faced with a single choice between a small–certain reward and a large–probabilistic reward of equal expected value (EV = probability*amount), people are typically risk-averse (Kahneman & Tversky, 1979). Over a wide range of probabilities ($1 > p > .1$), decision weight is less than the stated probability ($\pi < p$). For example, most people prefer $1,000 for sure to $2,000 with a stated probability of .5; the stated probability of the $2,000 would have to rise to about .6 or .7 before preference would reverse. Why, it may be asked, if discounting is so simple, is π not simply equal to the stated probability? The answer is that risk aversion, and the anomalies of decision making that stem from it, occur with stated, one-shot, probabilities. The prospects in Kahneman and Tversky's (1979) prospect theory are verbal questions such as: "Which would you prefer: $1,000 for sure, or $2,000 with a probability of .5?" People answering such questions usually prefer the sure thing over probabilistic alternatives of equal or even greater expected values. People's choices in such cases may be described by Equation 8 with $\varphi = \theta =$ odds against winning $= (1-p)/p$ (See Table 2):

$$\pi = \frac{1}{1 + k\theta^s} \quad (11)$$

When $k = s = 1$, $\pi = p$. As k or s increases, π decreases relative to p. However, when human participants expect stated probabilistic alternatives to be repeated, they typically prefer alternatives of higher expected value (Keren & Wagenaar,1987). Nonhuman subjects choosing among repeated probabilistic alternatives also come to exclusively prefer alternatives of higher expected value (Herrnstein & Loveland, 1975). These findings imply that, with repeated gambles, both k and s approach unity and $\pi \rightarrow p$. Moreover, "anomalies" such as the underweighting of base rates in Bayesian decision problems, are reduced or disappear entirely when decisions are repeated (Goodie & Fantino, 1996).

Rationality and preference reversals. With exponential delay discounting (Equation 6), two delay discount functions with the same time constant (interest rate i of Equation 6) would not cross (Ainslie, 1992). If, contrary to fact, exponential delay discounting described actual human and nonhuman choice among delayed rewards, and if $100 delayed by 10 days were preferred to $95 delayed by 9 days, then, after 9 days had passed, the $100, now delayed by a day, would still be preferred to the $95 available immediately. In other words, exponential delay discount functions for the $100 and the $95 would not cross. Nevertheless, in many instances, people do change their preferences over time, preferring the larger–later reward when both rewards are relatively distant, but switching their preference to the smaller–sooner reward when it becomes imminent. That is, unlike exponentials, actual discount functions may indeed cross.

Because exponential discount functions predict consistency of preference over time, economists call exponential discounting "rational." In economics, the reason why compound interest functions do not cross is that they are approximated by frequent recalculation of simple interest (compounding). For example, in the case of a savings account, your current balance would be incremented by a constant fraction at frequent periods between deposit and withdrawal. At any instant, all current balances (with the same interest rate) are increased by the same percentage of their current principal; thus, a lower balance can never catch up to a higher one on the basis of interest alone. (It is as if all employees of a company always received the same percentage raise regardless of their term of service or their performance. Under such conditions the rank order of their salaries would never change.)

Continuous compounding is rational behavior in economic situations where the magnitude of the discounting variable is not fixed in advance. You might take the money out of the bank at any time; you might cash in a bond;

[1] In the ideal case the energy source is a point. Since such a source is an idealization, the effective radius r is considered to originate at some distance from the surface of the source. Between the origin of r and the surface of the source, energy is considered to remain constant at E.

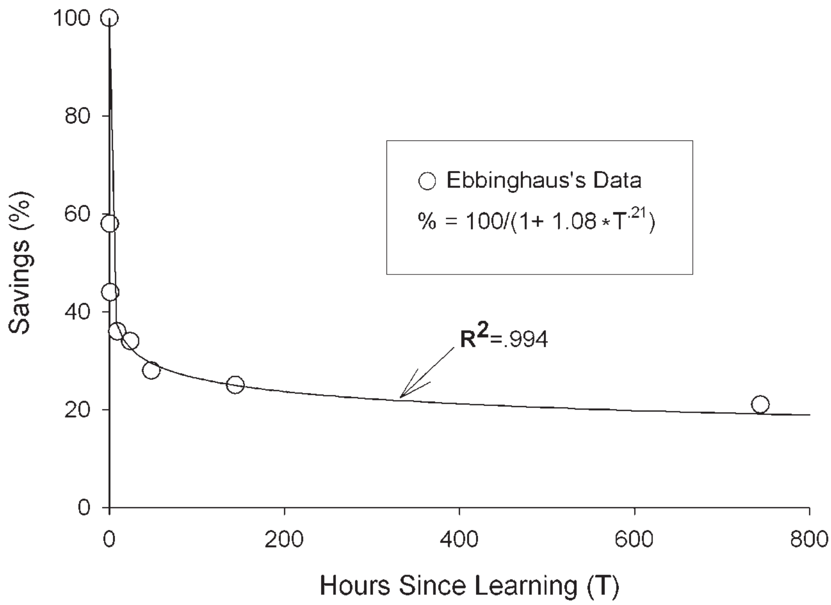

Figure 2. Ebbinghaus's forgetting function. Savings (learning time minus relearning time as a percentage of learning time) as a function of time between learning and relearning. The solid curve is the best-fitting exponentiated hyperbolic discount function (Equation 8).

you might (at least in theory) pay off your mortgage. However, what may be rational in economic interchange may not be rational in the case of an individual choosing between delayed rewards. In the case of exponential discounting of delayed rewards, the current value of the reward would be incremented continuously at a constant rate during the fixed delay. For $100 or $95 delayed by 10 or 9 days, respectively, the term (φ) is fixed. Thus, hyperbolic discounting (which, as we have seen, corresponds to simple interest) is appropriate and not in the least irrational. In other words, while it may be rational for a bank to approximate exponential discounting by compounding interest over the period of a loan when borrowing or lending money for an indefinite period, it is not necessarily rational for an individual to compound the appreciation of value continuously over fixed delays when choosing among delayed rewards.

If (in the above example of an individual choosing among rewards expected to be fixed in delay) after 9 days had passed, however, the new set of alternatives ($95 immediately versus $100 tomorrow) were unexpectedly offered, it would be irrational not to recalculate. The original offer, in that case, would be meaningless. There is nothing necessarily irrational about hyperbolic discounting per se, even by forward-looking organisms. That is, crossing functions are not in themselves necessarily irrational. The hyperbolic discount functions of two- and three-dimensional energy propagation would cross under corresponding conditions. (For example, the sound energy from a radio close to your ear may be more intense than that of a moving subway train 10 feet away but, stepping back 10 feet from the radio and 20 feet from the train, the sound energy intensities of the two would reverse.) But there is nothing irrational about this purely physical process.

What may be irrational, however, is a failure to account for a change of mind when it is known that a second choice will be offered and, on the basis of past experience, that one's own preference will reverse (O'Donoghue & Rabin, 1999). The alcoholic who vows, during a morning hangover, never to drink again, and then goes to a party, when in the past he has always gotten drunk at such parties, is behaving irrationally —not for his hyperbolic discount functions but for his failure to take his own future behavior into account.

Memory. Figure 2 shows Ebbinghaus's original forgetting function: savings (i.e., the time taken to learn a list of nonsense syllables minus the time taken to relearn the same list, divided by the time taken to learn the list originally) plotted as a function of the time between original learning and relearning (Ebbinghaus, 1885/1964, p. 76). The solid curve shows Equation 8 where x/X is the savings, and φ is hours between original learning and relearning. Equation 8 provides an excellent fit to the data (R^2 = .994).

Their hyperbolic form allows forgetting curves such as that of Figure 2 to cross. This reversal in memory is consistent with Jost's law which says that, ''…when two associations are of equal strength, a repetition strengthens the older more than the younger'' (Boring, 1957, p. 375). Because the associations are now at equal strength, the younger one, in order that it be strengthened less by a repetition, must be declining faster than the older one. That is, forgetting curves cross as time discount functions do. This in turn implies that their relative strengths reverse over time. Reversals of memory (the mirror image of preference reversals) are common in everyday life. For example, I remember yesterday's events better than those of my first day in graduate school. But, a year from now, yesterday's events will have vanished almost to oblivion while those of my first day in graduate school, although only vaguely recalled, will still be discernable. Again, there is nothing irrational about crossing functions, whether they occur in physics or in psychology.

Matching and discounting. The time-allocation version of Herrnstein's (1961) matching law says:

$$\frac{T_1}{T_1 + T_2} = \frac{R_1}{R_1 + R_2} \tag{12}$$

That is, given two alternative activities (T_1 and T_2) between which time is allocated, the proportion of time allocated to one of them $[T_1/(T_1 + T_2)]$ equals the proportion of obtained reinforcers $[R_1/(R_1 + R_2)]$ contingent on that activity. Our object here is to transform Equation 12 into a discount function—an equation in which a variable is expressed as a function of its own maximum value.

Equation 12 implies that the time allocated to each activity is discounted by reinforcement contingent on the other activity. As R_2 increases, T_1 decreases (and vice-versa). As $R_2 \to \infty$, $T_1 \to 0$. When $R_2 = 0$, T_1 is maximal. Let us represent this maximal value as T^*. Keeping all other sources of reinforcement constant, $T^* = T_1 + T_2$. In terms of Equation 1, X would be T^*, x would be T_1, and π would be $R_1/(R_1+R_2)$:

$$\frac{T_1}{T^*} = \frac{R_1}{R_1 + R_2} \tag{12a}$$

The matching law says, in other words, that the time allocated to an activity is discounted by reinforcement contingent on an alternative activity. The common inverse relationship in matching and hyperbolic discounting was noted in the earliest hyperbolic discounting formulations (Chung & Herrnstein, 1967). In Equation 2, the probabilistic variable may be transformed into a magnitude by solving for δ and substituting $R_1/(R_1+R_2)$ for π:

$$\delta = \frac{1 - \pi}{\pi} = \frac{1 - \frac{R_1}{R_1 + R_2}}{\frac{R_1}{R_1 + R_2}} = \frac{R_2}{R_1} \tag{13}$$

In Equation 4:

$$\frac{x}{X} = \frac{1}{1 + \frac{R_2}{R_1}} \tag{14}$$

and:

$$\frac{T_1}{T^*} = \frac{1}{1 + \frac{R_2}{R_1}} \tag{15}$$

The magnitude, R_2/R_1, varies from 0 to ∞ as the fraction, $R_1/(R_1+R_2)$, varies from 1 to 0. As R_2/R_1 increases from 0, T_1 decreases monotonically from T^* and approaches 0. Equation 15, a form of simple matching, is thus also a hyperbolic discount function. In comparison to Equation 5, the fraction R_2/R_1 may be the discounting variable with $k = 1$; or R_1 may the discounting variable with $k = 1/R_2$; or R_2 may be the discounting variable with $k = R_1$.

Similar reasoning may be applied to the generalized matching law (Baum, 1974). Equation 12, expressing the equality of two fractions, may be transformed into an expression of the equality of two ratios by cross-multiplication and division:

$$\frac{T_1}{T_2} = \frac{R_1}{R_2} \tag{16}$$

Equation 16 is mathematically equivalent to Equation 12 (simple matching). In Baum's generalization:

$$\frac{T_1}{T_2} = b\left(\frac{R_1}{R_2}\right)^s \tag{17}$$

where b stands for *bias* and s stands for *sensitivity*. The constant b ($b \geq 0$) measures bias due to factors other than number of obtained reinforcers: for example, the attractiveness of one or another choice response itself, or the influence of other variables such as differing amounts or delays of individual reinforcers. Bias may vary from $b = 0$ (complete preference for alternative 2), through $b = 1$ (indifference to factors other than obtained reinforcers), to $b \to \infty$ (complete preference for alternative 1). The constant s ($s \geq 0$) measures sensitivity of choice behavior (the left side of Equation 17) to reinforcement (the right side). When $s = 0$, T_1/T_2 is completely insensitive to variation in R_1/R_2. When $0 < s < 1$, T_1/T_2 "undermatches" R_1/R_2. When $s = 1$ and $b = 1$, T_1/T_2 simply matches R_1/R_2. When $s = 1$ and $b \neq 1$, T_1/T_2 is simply biased towards one or the other alternative. When $s > 1$, T_1/T_2 "overmatches" R_1/R_2.

To convert Equation 17 to a discounting equation corresponding to Equation 15, the left side of Equation 17 must look like the left side of Equation 15. Inverting and adding 1.0 to both sides of Equation 17:

$$\frac{T_2}{T_1} + \frac{T_1}{T_1} = \frac{R_2^s}{bR_1^s} + \frac{bR_1^s}{bR_1^s}$$

$$\frac{T_1 + T_2}{T_1} = \frac{bR_1^s + R_2^s}{bR_1^s}$$

Inverting again:

$$\frac{T_1}{T_1 + T_2} = \frac{bR_1^s}{bR_1^s + R_2^s}$$

$$\frac{T_1}{T_1 + T_2} = \frac{1}{1 + \frac{R_2^s}{bR_1^s}}$$

Setting $T^* = T_1 + T_2$:

$$\frac{T_1}{T^*} = \frac{1}{1 + \frac{1}{b}\left(\frac{R_2}{R_1}\right)^s} \tag{18}$$

Equation 18 shows generalized matching in the form of a discount equation. (See McDowell, 2005, Equation 6, for a corresponding reformulation of generalized matching in another context.) The discounting variable is the ratio R_2/R_1 as in simple matching.

Utility. Because matching is, arguably, basically a form of maximizing (Rachlin, Green, Kagel, & Battalio, 1976), and as shown in the previous section, it also is a kind of discounting, it may be worth looking at utility functions also as discount functions (that is, functions of their maximum value). Suppose a consumer is allocating a budget of $\$N$ between two commodities, c and d. The constraints are as follows: the price of c is $\$p$ per unit, and the price of d is $\$q$ per unit; the numbers of units bought of the two commodities are C and D; then:

$$N = pC + qD \tag{19}$$

Given these constraints, the maximum number of units of c that can be bought (C^*) is N/p, whereas the maximum number of units of d that can be bought (D^*) is N/q. A common utility function, assuming constant elasticity of substitution (Kagel, Battalio, & Green, 1995), is:

$$U = V_c C^m + V_d D^m \tag{20}$$

where U is utility, and V_c and V_d are weighting factors representing the value of a unit of each commodity. The value of a commodity unit may in turn be a function of various factors such as amount, rate of consumption, delay, etc., as well as the inherent quality of the individual commodity. The exponent m is the key economic variable, *substitutability*. Unlike s in Equation 8, which is simply a positive number ($s \geq 0$), m is equal to or less than unity and may be negative ($m \leq 1$).[2] When $m = 1$, c and d are perfectly substitutable (like Coke and Pepsi; the consumer may prefer one to the other but doesn't particularly value a mixture over either cola drink alone). As m decreases towards 0, c and d are less than completely substitutable (like apples and

[2] In certain special cases, s may be negative. For example, in choosing between two commodities that are complements, such as food versus water, decreasing the rate of either commodity would increase the time allocated to obtaining it, resulting in negative values of s; this result is called *antimatching* (Rachlin, Battalio, Kagel, & Green, 1981). In the vast majority of studies of matching, however, the objects of choice are identical reinforcers (food versus food). In such cases, s is always positive.

pears; the consumer would like to have a mixture). When m is negative, c and d are complements (like beer and pretzels; the value of each is enhanced by the other). As m grows more negative, c and d grow more complementary (like left shoes and right shoes; a mixture becomes necessary to attain any value at all). It is lack of complete substitutability that generates trade (i.e., economic activity). If one person has all the apples and the other has all the pears, both people will gain in utility if they trade.

Given the constraints of Equation 19, and assuming a utility function of the form of Equation 20, it can be shown (see Appendix) that the ratio of C (number of units of c bought) to C^* (maximum number of units of c that can be bought given the constraints) at the point of maximum utility is given by:

$$\frac{C}{C^*} = \frac{1}{1 + \left(\frac{p}{q}\right)^{\sigma-1}\left(\frac{V_d}{V_c}\right)^{\sigma}} \tag{21}$$

where $\sigma = 1/(1-m)$. As the commodities approach complete substitutability ($m \rightarrow 1$), the exponent σ increases indefinitely ($\sigma \rightarrow \infty$). At this point, with the ratio schedules implied by fixed commodity prices, choice is all-or-none for one of the alternatives depending on value and price. As the commodities become less substitutable ($m \rightarrow 0$), $\sigma \rightarrow 1$. As the commodities become more and more complementary ($m \rightarrow -\infty$), $\sigma \rightarrow 0$. Thus, within the limits of m, σ (like s in Equation 18) is a positive number.

Note the similarity of Equation 21 to Equation 18. The discounting variable V_d/V_c corresponds to the ratio of reinforcement rates in the generalized matching law, and the ratio of the prices of the commodities, raised to the power $\sigma-1$, acts as bias. In both equations, it is the relative value of the alternatives and their prices, not their absolute values, that determines the degree of discounting and that constitutes the fundamental independent variable in studies of choice. The exponents s and σ determine sensitivity of the dependent variable to the independent variable, but in Equation 21 that sensitivity is shown to depend, in turn, on the substitutability of the alternatives.[3]

Conclusion. Many phenomena in psychology as well as in economics and physics may be viewed as discounting phenomena, and when they are so viewed the phenomena tend to take on similar hyperbolic forms. The fact that molar processes such as matching and utility maximization may be viewed as discounting processes indicates that discounting need not be a strictly molecular process with unitary discounting variables such as time or distance, but rather may be a molar process with compound molar discounting variables such as ratios of reinforcement rates, commodity values, and commodity prices.

[3] The assumption that each commodity has a single price is equivalent to an assumption of concurrent fixed-ratio schedules in operant choice experiments. Under such conditions, with the usual operant alternatives of identical reinforcers differing only in rate, all-or-none preference for the higher rate is typically found (Green, Rachlin, & Hanson, 1983; Herrnstein & Loveland, 1975). Equation 21 makes this prediction. Concurrent-interval schedules, with which matching is usually studied, impose a complex relationship between price and ongoing choice. For an analysis of this more complex situation see Rachlin (1982).

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Received September 28, 2005
Final acceptance March 4, 2006

APPENDIX

Derivation of a Discount Function from Utility Maximization

From Equation 19:

$$D = \frac{N - pC}{q}$$

Substituting for D in Equation 20:

$$U = V_c C^m + V_d \left[\frac{N - pC}{q}\right]^m$$

Differentiating and setting the derivative equal to zero to determine the maximum value of U:

$$\frac{dU}{dC} = mV_c C^{m-1} - \frac{mpV_d}{q}\left[\frac{N - pC}{q}\right]^{m-1} = 0$$

Solving for C:

$$C = \frac{N/p}{1 + \left(\frac{p}{q}\right)^{\frac{m}{1-m}} \left(\frac{V_d}{V_c}\right)^{\frac{1}{1-m}}}$$

At $D = 0$, $C = N/p = C^*$ (all of the budget is spent on c). And setting $\sigma = 1/(1-m)$:

$$\frac{C}{C^*} = \frac{1}{1 + \left(\frac{p}{q}\right)^{\sigma-1} \left(\frac{V_d}{V_c}\right)^{\sigma}}$$